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Article Submission

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Introduction

This project started out when Nat Friedman asked Robert Krawczyk to design a computer generated version of the aluminum triple twist Mobius band in Figure 1(a). The first version is shown in Figure 1(b). The digital version has all of the same geometric aspects of the aluminum sculpture; the corner connections and the variable change in speed of the twist itself. After some discussion, there followed a whole series of sculptures with twists which we refer to as twistors for short. Some, but not all, twistors are one-sided. Another triple twist Mobius is shown in Figure 2. Here the twist has no straight parts on sides.

(a) Aluminum, 9 x 9 x 2 inches.  
(b) Computer Generated

Figure 1. Physical and computer generated versions of Triple twist Mobius band 1.

Figure 2. Two views of Triple twist Mobius band 2.

Figure 3. Two views of Triple twist Mobius band 3.
A third version is shown in Figure 3, where the twist is enlarged and more pronounced. From a mathematical viewpoint, a twist is just a twist. From a sculptural viewpoint, there are variously shaped twists, such as long and slow, tight and fast, etc. and we will consider various twist shapes below.

The next twistor is a five-twist Mobius band, as shown in Figure 4. We stayed with the triangular arrangement with twists of two sizes for variety, rather than a pentagon with five equal size twists.

A seven-twist Mobius band is shown in Figure 5 with twists of two sizes. Also note that the top twists are even tighter than those in Figure 4.

Lastly, a nine-twist Mobius band is shown in Figure 6. Here the twists are the same size.
Divided Bands

It is well known that a divided Mobius band remains as one single two-sided band. The twistor in Figure 1 is shown divided in Figure 7. The shape of the dividing space matches the shape of the sculpture, hence the space is a triple twist Mobius space. In Figure 7(c) the ends are joined for support as, for example, in the case of a large outdoor sculpture.

Figure 7. Three views of Triple Twist Mobius Band 1 divided.

In Figure 8 are views of the divided nine-twist Mobius Band 1, where a more supportive join is shown in the ends in Figure 8(c). The space is a nine-twist Mobius space.

Figure 8. Three views of Nine-twist Mobius Band 1 divided.
**Even Number of Twists**

We have also experimented with even number of twists. For even number of twists, the twistors are two-sided and the divided twistor consists of two pieces.

A twistor with six twists (six twistor) is shown in Figure 9.

A divided twistor with six twists is shown with just the ends joined in Figure 10.

An alternate six twist is shown in Figure 11, where quarter twists are on each side of a half twist.

![Figure 9. Two views of Six Twistor 1.](image)

![Figure 10. Two views of Six Twistor 1 divided.](image)

![Figure 11. Two views of Alternate Six Twistor.](image)
Mirrored Twistors

The twists in Figure 1(a) are considered right-handed and the twists in Figure 1(b,c) are left-handed. One is the mirror image of the other. When the two mirror images are combined, we refer to this as a Mirrored Twistor. The mirrored twistor corresponding to the twistor in Figure 3 is shown in Figure 12. The mirrored twistor has self-intersections of the component twistors.

Figure 12. Two views of Mirrored Triple Twist Mobius 3.

Figure 13. Two views of Mirrored Seven twist Mobius Band 1.

The mirrored version of the seven twist Mobius band in Figure 5 is shown in Figure 13. The divided mirrored triple twist Mobius band 1 is shown in Figure 14.

Figure 14. Two views of Divided mirrored triple twist Mobius Band 1.
Rotation Combination

The sculpture in Figure 15 is obtained by rotating the twistor in Figure 14 a quarter turn and then combining the rotation with itself. A top view is also shown.

![Side View](a) Side View  
*Figure 15. Two views of rotation combination of divided mirrored triple twist Mobius Band 1.*

![Top View](b) Top View

The top view in Figure 15 shows that the sculpture can again be rotated a quarter turn and then combined with itself resulting in the sculpture in Figure 16.

![Side View](a) Side View  
*Figure 16. Two views of double rotation combination of divided mirrored triple twist Mobius Band 1.*
Three-way Division

If a Mobius band is divided into thirds, then the middle third is a separate Mobius band and the two outer thirds form one two-sided band as in the case of cutting a Mobius band in half. This is shown in Figure 17 for the twistor in Figure 1. The ends are joined for support since the middle band is separated from the outer joined bands.

Figure 17. Two views of Triple twist Mobius band 1 divided in three.

Figure 18. Two views of Nine twist Mobius Band 1 divided in three.

The nine twistor 1 divided in three is shown in Figure 18. The corresponding mirrored twistor is shown in Figure 19.

Figure 19. Two views of Nine twist Mobius Band 1 mirrored divided in three.
Rotation in the Paper Plane

An image can be rotated in the paper plane and combined with the image as shown in Figure 20, where the image is six twist 1 mirrored.

Figure 20. Two views of Six twist 1 mirrored rotated in plane and combined.

In Figure 21 six twist 1 mirrored is rotated a quarter turn left and right and the two rotations are combined.

Figure 21. Two views of Six twist 1 mirrored with right and left quarter turns combined.
**Vertical Mirror Image In the Paper Plane**

We consider starting with divided six twistor 1 in Figure 22. A quarter turn rotation of Six twistor 1 divided is now combined with itself, as in Figure 23.

![Figure 22. Two views of Six twistor 1 divided.](image)

![Figure 23. Two views of Quarter turn rotation of Six twistor 1 divided combined with itself.](image)

The twistor in Figure 23 is now combined with an eighth turn rotation of itself and then mirror imaged to obtain the twistor in Figure 24. The twistor in Figure 24 is now vertically mirror imaged in the paper plane to obtain the sculpture in Figure 25.

![Figure 24. Two views of Rotation combination and mirror image.](image)

![Figure 25. Vertical mirror image in the paper plane.](image)
Knotted Twistors — Twistor-Knots
Carlo H. Séquin
EECS Computer Science Division, U.C. Berkeley
sequin@cs.berkeley.edu

Figure 1: “Twistors” real and virtual by Nathaniel Friedman and Robert Krawczyk.

The creations discussed in this paper were inspired by the Twistor sculpture (Fig.1a) that Nathaniel Friedman submitted to the Art Exhibit of the Joint Mathematics Meeting to be held in San Antonio, Texas, in January 2015, and by the ongoing joint development of many related forms (Fig.1b–e) together with Robert Krawczyk [2]. The idea behind this extension was simple: Rather than extruding the twisted bands along the edges of one or more triangles, let’s try to fit them along the various beams that make up a sticks representation of some simple and symmetrical mathematical knots. Then try to find out how densely the various twisted beams can be packed when the twisting is optimally adjusted so that the various helical forms nicely interlock without creating any beam intersections.

Trefoil Twistor-Knot based on Knot 3_1

The first candidate is the knot 3_1, the trefoil knot (Fig.2a). Figures 2b–d show three different tight sticks versions of this knot, where every tube just touches its neighbors. Among the three configurations shown, Figure 2c is the one with the minimal overall tube length. I use this geometry to form a first Twistor-Knot by running a twisted band inside every tube (Fig.2e).

Figure 2: (a) Smooth trefoil knot; (b–d) various symmetrical, tight sticks realizations, (c) has the overall shortest tube length; (e) a twisted band run along the path of (c).

Starting with the tightest sticks configurations (Fig.2c), twisted bands of width $w=2$ will readily fit into the normalized tightly packed tubes of radius one. Thus we just need to adjust the azimuth and
twist angles for the two types of beams (the inner three and the outer three). The cleanest results are obtained if the six outer seams, where a pair of twisted bands meet, are oriented perpendicularly to the plane defined by the two beams ending in that seam. For each type of beam, we adjust its azimuth angle (rotation around the axis of the beam), so that the start of the beam lines up with the orientation of that normal vector; next we adjust the beam’s twist so that the end of the beam lines up with the plane normal at the far end.

However, we still can make some choices about these two twist values and change them in increments of 180°. Figure 3a, shows a configuration based on the tight trefoil with beams of a nominal width \( w=2 \), where the inner beams are experiencing a half twist, and the outer bands execute a full twist. This yields a graceful Twistor-Knot that is not packed too tightly. In Figure 3b the direction of twisting has been reversed. This allows to broaden the width of the bands by a factor of 1.4, without occurring any band intersections – which in the arrangement of Figure 3a, would first occur near the center. The double half-flip of the outer beams also comes in handy: they arch neatly over the maximal radial extension of the inner bands at their mid-points. However, this result looks somewhat too crowded. In addition the difference of the actual amount of twisting of the inner bands as compared to those of the outer bands has increased, because of the way that the normal vectors at the ends of the beams are tilted with respect to one another.

![Figure 3: Twistor-Knots 3_1: (a) Tight configuration with a half twist on the inner beams and a full twist on the outer beams; (b) a denser configuration with twists in the opposite orientation.](image)

The reader may wonder whether one could combine the inner twisting of Figure 3b, which is optimal to prevent intersections of those bands near the center, with some other more gentle twist on the outer beams. It turns out that using opposite twisting in the two types of beams is not advisable: the pairs of beam joining in the outer seams would jam into one another in an unattractive way. On the other hand, giving the outer band a minimal amount of twist, just enough to make up for the different orientations of the seams at the two ends, results in a simple, but pleasing configuration. (Even the configuration with no effective twist in any of the bands looks somewhat attractive! – see Figure 2e.)

For this Twistor Knot I designed two special textures to be applied to the beams in the above virtual renderings. They are both of dark blue color at the ends of the individual beams and get brighter towards the middle of the beam. This symbolically represents the way that beams get
fabricated for a metal sculpture like Figure 1a: The ends are held in heavy clamps or in a vice; the exposed middle is heated to make the metal soft, and then the desired twist is introduced into the beam.

**Pentafoil Twistor-Knot based on Knot 5_1**

For the Twistor-Knot 5-1 I took a different approach. Rather than starting with a sticks configuration of knot 5_1 (Fig.4a), I started from the planar Pentagram shown in Figure 4b. To avoid creating intersections, I had to introduce some curvature into the five beams. They are bending out of the symmetry plane to form the needed over/under-crossings; they also bend laterally in this plane to place these crossings into optimal locations (Fig.4c).

![Figures 4a, 4b, 4c, 4d](image)

*Figure 4: (a) Smooth pentafoil knot; (b) 2D pentagram; (c) pentagram with curved bands; (d) legs for “Pentafoil Knot Table” [1].*

Each ribbon makes a double-flip, so that is in a mostly horizontal orientation at the locations of the crossings (Fig.5). The sweep path of each beam is specified with a quintic (6th-order) Bézier curve. The four inner control-points allow me to fix the tangent direction in which the beams take off from the five outer junctions and to control the elevation at the over/under-passes. They also allow me to adjust the rate of twisting in a non-linear manner; I pushed the highest twist-rate to the center of the beam, so that the ribbons have the optimal orientation where they cross.

![Figures 5a, 5b](image)

*Figure 5: Twistor-Knot 5_1: (a) view from the top, and (b) view from the side.*
Again the issue arises how to orient in space the five seams where two ribbons join. For simplicity, I have placed them normal to the symmetry plane, as indicated by the five legs of the “Pentafoil Knot Table” [1] shown in Figure 4d. This seems justified, since I can keep the initial tangent directions of all the sweep paths within this plane. Figure 5 shows the resulting Pentafoil Twistor-Knot from “the top”, as well as from the side, to give a better understanding of the 3-dimensional bending of the beams. In this model I used a continuous rainbow texture to make it easy to track the flow of the ribbon around the pentagram loop.

**Figure-8 Twistor-Knot based on Knot 4_1**

The third knot studied offers a new challenge. The previous two knots have some obvious chirality, and they exhibit D3 and D5-symmetry, respectively. Knot 4_1 is amphichiral, i.e., it can be deformed into its own mirror image. In some renderings this is more obvious than in the standard representation (Fig.6a). Figure 6b exhibits D2-symmetry but does not hint at the amphichiral quality of this knot. On the other hand, the configuration shown in Figures 6c–e all have 4-fold glide symmetry (S4), where a rotation by 90° around the z-axis and simultaneous mirroring of the z-coordinates will bring the transformed copy into coincidence with its original shape. Figure 6d shows a reasonably tight sticks configuration, and this serves as a starting point for the minimally twisted ribbon version shown in Figure 6e, in which the red and blue beams have been given some slant to make the loops more balanced and the overall shape less stiff.

![Figure 6](image)

Figure 6: Figure-8 Knot: (a) Smooth rendering; (b) D2-symmetric diagram; (c) S4-symmetric 3D model, (d) 8-sticks model, (e) minimally-twisting ribbon knot, – also with S4-symmetry.

Starting with the configuration of Figure 6e we can select two different amounts of twist in the “horizontal” (red, blue) and “vertical” (orange, cyan) beams. Because of the S4 symmetry the twists are equal and opposite for both color pairs; thus the net overall twist must be zero, as is required for an amphichiral knot. Two resulting designs are shown in Figures 7a,b; their twist values are listed in the figure caption.

**References**


Figure 7: Twistor-Knots 4_1: (a) Twist: hor: ±155°, ver: ±251°; (b) Twist: hor: ±338°, ver: ±430°.
Haresh Lalvani: New Methods-New Sculpture.
Nathaniel Friedman
Professor Emeritus Mathematics, SUNY-Albany
nat.isama77@gmail.com

Introduction

Haresh Lalvani is Professor of Architecture at Pratt Institute in New York City. For the past seventeen years he has been collaborating with the renowned art-metal fabricator Milgo-Bufkin creating unique sculptures. The main concept is to start with metal in a two-dimensional form and create a three-dimensional form. One of his morphological quests is understanding how force shapes matter, another is to understand the origin of form.

We will first show his use of sheet steel to form columns and other surfaces, works that began in 1997. In 2004 a set of 4 nine foot architectural columns were commissioned for acquisition by the Museum of Modern Art (MOMA) in New York City for the opening of their new building in the Fall of that year. These columns are part of their permanent collection. They were formed from a single sheet of titanium using an algorithmic process, possibly the first time titanium was folded that way.
In 2007 he began inventing original material in the form of sheet metal with laser-cut out spaces. This material was used to form various spheres and ellipsoids, referred to as hypersurfaces below, an extension of his earlier laser-cut paperboard spheres displayed at the Cathedral of St John the Divine, NYC, in 1996, where he was an Artist-in-Residence. In 2006 he began the Xurf art series on larger scale based on his new invention of expandable surfaces (in 1998) of a different type of perforated metal that was based on geometric tiling patterns. Also included in the Xurf series was perforated mirrored stainless steel where the pattern of perforations was irregular. In mirrored stainless steel, this resulted in faceted surfaces with striking reflected images that were very reminiscent of Cubist paintings. In 2009 came the Purf series, where he introduced another new stainless steel screen material to generate three-dimensional forms, including large sculptures and architectural enclosures, all formed from a continuous metal sheet.

To summarize, Lalvani first applied force to bend sheet metal and generate a variety of architectural columns. He next invented sheets of perforated metal and the different perforations then allowed the metal to be forced into various shapes. The force applied to the perforated metal sheet results in bending and spaces opening up where the material is stretched so that one can visualize the force acting to generate a large variety of three-dimensional forms.

Columns.

Haresh Lalvani used an algorithmic process to create architectural columns from sheet metal. A procedure was developed so that single metal sheets are shaped by computer driven equipment according to algorithmically generated geometries. The process allowed for a variety of curvilinear columns to be created. In Figure 1, Haresh is shown with several columns of varying height. This was his first experiment to generate infinite shapes in one material using one algorithm and one method of forming.

In 2004, the Museum of Modern Art (MOMA) in NYC commissioned his set of 4 titanium columns (Figure 2) for the opening of their new building. MOMA acquired these columns for their permanent collection.

Hypersurfaces: Pods and Seeds.

Columns are formed from sheet steel and stand on the floor. By contrast, Pods and Seeds are closed round objects with a spherical geometry. They are based on Seeds and Pods occurring in nature, a long-term interest of his in how nature transmits genetic information. Pods and Seeds are first created as 3D models on the computer, then converted into 2D sheet material for fabrication, then folded into 3D. Sometimes paper models are made for study.
An object with closed spherical geometry cannot be formed from a single sheet without allowing for folding. The simplest case is a tetrahedron which can be folded from a pattern of four connected equilateral triangles. The remaining four Platonic solids are the cube, octahedron, dodecahedron, and icosohedron, which also can be folded from a pattern of squares for a cube, pentagons for a dodecahedron, and equilateral triangles for an octahedron and icosohedron. Pods will have surface patterns of fold lines that allow the steel to be formed into a closed sphere-like shape. Lalvani’s patterns for fold lines will consist of various parts that are much more complex compared with the Platonic solids. The variety in the pattern of fold lines necessary for the process of forming the Pods are also of major artistic and design interest.

![Figure 3. (a) Constellation, detail.](image1)

![Figure 3. (b) Beans, detail.](image2)

![Figure 4. (a) Constellation, painted steel, 24" dia., 2007.](image3)

![Figure 4. (b) Holeysphere, painted steel, 24" dia., 2007.](image4)

![Figure 4. (c) Pod, painted stainless steel, 12" dia., 2009.](image5)

![Figure 4. (d) Beans, painted stainless steel, 48" dia., 2009.](image6)

Examples of Pods are shown in Figure 4(a-d). Each Pod consists of steel that has a pattern of polygons with possible fold lines. Windows (spaces) are then laser cut out of the material that may or may not cut across some of the fold lines. The windows allow one to see into and through the sculpture. This already makes the Pods interesting because they are not the typical continuous closed surfaces with no windows.

Here we will pause to introduce the concept of hyperseeing because the Pods in Figure 4 allow one to hypersee a sculpture. In order to see a two-dimensional painting on a wall from one viewpoint,
one must step back from the wall in the third dimension of the room. The painting can then be seen completely. From one viewpoint one can see the outline of the shape of the painting (generally rectangular) as well as every point in the painting. To see a three-dimensional object such as a sculpture or a building completely from one viewpoint, theoretically one would have to step back in a fourth spatial dimension. From one viewpoint, one could then theoretically see every point on the surface of the object as well as every point within the object (x-ray seeing). Four-dimensional space is referred to as hyperspace and seeing in hyperspace we have referred to as hyperseeing. Thus in hyperspace one could hypersee a three-dimensional object completely from one viewpoint. In our world, we can think of hyperseeing a sculpture by walking around it to see it from multiple viewpoints. We can think of hyperseeing a building by also walking around inside the building to experience being in the interior spaces of the building. In general, we can think of hyperseeing A from the viewpoint of B for general A and B. As an example, we can view art from the viewpoint of mathematics and vice versa. This was the motivation for naming this publication Hyperseeing.

Figure 5. Detail of Holeysphere.

Cubist painting, that is, seeing from multiple viewpoints, is an example of hyperseeing. Picasso would paint a nose in profile and an eye from the front. Lalvani’s mirrored pieces in figures 22 and 26 below show multiply fractured images and exemplify hyperseeing from multiple viewpoints.
Pods and Seeds are examples of hypersurfaces because the patterns of fold lines are three-dimensional projections of higher dimensional structures as we will see with Alien discussed below. Since the surface patterns of fold lines are projections of higher dimensional structures, Lalvani’s geometries are new combinations of the basic flat, spherical, and hyperbolic (saddle-shaped) geometries.

Constellation (Figure 4(a)) has windows, we can see into, as well as see through the space to view the opposite windows. We can hypersee Constellation by walking around it as well as looking through it. A fly could fly through and around to experience the space of Constellation.

Experiencing the space of a sculpture is the essence of Richard Serra’s large steel walls and shells. One not only walks around the outside of the sculpture but walks into the space of the sculpture. Experiencing space is central to architecture and will be discussed again later when we consider Purf.

A detail of Constellation is shown in Figure 3(a). The fold lines form a variety of triangles. The windows, inside the triangles formed by the fold lines, are shaped as rounded triangles. The triangles group in variously shaped parallelograms, pentagons, hexagons, heptagons, etc. This is the advantage of digital design which enables infinite variety within certain parameters just as in nature. No two persons are exactly alike and the same can be said for snowflakes, trees, leaves, and nature in general. Infinite variety is a major aspect of Lalvani’s designs. Another critical point is that the pattern of rounded triangles does NOT repeat. This is because the pattern is derived from higher dimensional polyhedra created by Lalvani that are similar to the two-dimensional Penrose tiles which do not repeat. Lalvani’s surface design is highly sophisticated.

A detail of HoleySphere in Figure 4(b) is shown in Figure 5. Each window is inside a parallelogram but cuts across a diagonal. The window space also folds because the window is on a fold line. We can think of this as two-dimensional window space folding. The parallelograms are of various sizes and the windows are variously sized circles and ellipses. An infinite variety within certain parameters. As in Constellation, the windows allow one to hypersee patterns and light effects on the other side.

In Pod (Figure 4(c)) the windows cut across fold lines and the two-dimensional spaces of the windows fold as well. The windows of Pod are relatively large compared to Constellation and HoleySphere and the patterns on the opposite side show up better as if framed by the windows. One has the hyperseeing feeling of experiencing both sides at once.

Beans in Figure 4(d) is much larger than Pod. A detail image of Beans is shown in Figure 3(b). As in Pod, the windows cut across fold lines and therefore bend. The shapes of the windows in Beans are similar to those in Pod and display the same infinite variety.
Alien

Alien is the vertical pod shown in Figure 6 (a). The windows in Alien are circles and ellipses and occur within fold lines or are bisected by fold lines which bend the windows. Here too, the windows allow one to see the inner space and patterns on the opposite side. In the surface detail in (b) there are lighted regions where one can see rectangles and parallelograms that are actually three-dimensional projections of higher dimensional structures.

Seeds

Another group of sculptures, Seeds, are shown Figure 8 (a-c). Although the shapes are similar to Pods, they are constructed from discrete partially circular stainless steel elements. The elements either join along an arc or just tangentially. The dashed lines indicate where the metal can slightly fold down. The material allows for various closed shapes such as the sphere in (a) and the more oblate forms in (b,c). The dashed lines allow for variable fold as in (c) on the “equator” where the fold is fairly sharp. On other shapes there is hardly any fold where the surface is only slightly bent.
Thus here we see how Lalvani designed a relatively simple element that can generate closed surfaces with variable shapes.

**Figure 7. Seed54, stainless steel, 8ft H, 2012 and detail.**

**Seed54**

Haresh Lalvani’s stainless steel sculpture *Seed54* in Figure 7 was commissioned by RXR Reality and is located at 1330 Avenue of the Americas in NYC. It has a shape similar to *Alien* but is larger and otherwise quite different. The windows are kidney shaped and cross over fold lines so all the windows bend. Being outdoors, it interacts visually with sunlight falling on the stainless steel surface. Shadows are created from the patterns of the windows which makes it interesting to see patterns of light and shadow on the opposite side. The infinite variety of the surface design makes it interesting from all viewpoints. Furthermore, as the angle of the sun changes, all the reflected patterns change. It is alive.

**Xurf**

The first example of *Xurf* art piece shown here is perforated sheet steel that consists of hexagons connected by equilateral triangles at each corner, as shown in Figure 9. The perforations allow the material to stretch in different directions to open up the spaces between hexagons and triangles. *Thus the pattern varies as the modulated surface changes.* A kidney shaped form in this
expandable material is shown in Figure 10(a). A second kidney shaped form is shown in Figure 10(b).

Forms like Eros in Figure 11 (and Whorl, Fig.21) are wall sculptures with curved outlines somewhat reminiscent of works of Jean Arp. However, Arp’s works were flat whereas here the modulation of the surface is of main significance. Another important feature of Lalvani’s work is the reflection of light. Each hexagon is a source of directional light.

A modulated disk consisting of hexagons and triangles is shown in Figure 11. This is a minimal elegant form reminiscent of Brancusi with a reflective surface replacing a polished a surface. When
Lalvani first saw *Eros* shape itself under force, it reminded him of the work of Jackson Pollack whose drip paintings were similarly process-driven. In his early experiments in self-shaping of *Xurf* pieces, he also sees an affinity with Einstein’s idea that mass bends space. His pieces are bent by force and the warped space is the shape of space the metal surface traces as it goes from flat to curved.

![Sunburst](image1.png)  ![Starburst](image2.png)

*Figure 13. Sunburst, stainless steel, 2009, 70W x 48H x 7D inches.*  *Figure 14. Starburst, stainless steel, 2009, 70W x 36H x 7D inches.*

![Square Squared](image3.png)  ![Nebula](image4.png)

*Figure 15. Square Squared, stainless steel, 2007, 30 Dia x 7D, inches.*  *Figure 16. Nebula, stainless steel, 48 Dia x 9D inches.*

The second example is perforated steel consisting of squares joined at corners shown in Figure 12. This is a detail image of the large work *Sunburst* in Figure 13. *Sunburst* consists of individual sections with a variety of surface forms. As in *Alien*, one can see projections of cubes in *Sunburst*. Reflection of light is again a main feature.
A related sculpture *Starburst* is shown in Figure 14. These two sculptures definitely have perimeters indicative of bursting. In the disk *Square Squared* in Figure 15 the perforations consist of large squares connected by small squares. The same perforation pattern occurs in the larger disk *Nebula* in Figure 16.

The image in Figure 18 is a detail of *Amun* shown in Figure 17. Here large equilateral triangles are connected by smaller hexagons. In the detail image one can clearly see how the spaces open up when the material stretches.

![Figure 17-18. Amun, stainless steel, 2008, 48Dia x 9D inches and Amun detail.](image)

The image in Figure 20 is a detail of *Phobos* shown in Figure 19. This is a completely different perforation pattern. Here the spaces have a curvy shape of their own.

![Figure 20. Phobos, Stainless Steel, 2008, 48Dia x 9D inches and Phobos detail.](image)
The perforation patterns in the works considered thus far relate to orderly tiling patterns. Lalvani has also introduced much more irregular perforation patterns as seen in *Whorl* in Figure 21. This is a very fractured image reminiscent of a Cubist painting.

**Figure 21. Whorl, stainless steel, 2010, 48 Dia x 9D inches.**

Lalvani also introduced mirrored stainless steel which resulted in striking Cubist-like images as shown in Figures 22-26.

**Figure 22. Portrait of Deborah Buck, mirrored stainless steel.**  
**Figure 23. Xurf Cubist Portrait of Haresh Lalvani.**
The *Purf* sculptures are formed from a steel screen material invented by Lalvani that is quite different from his previous expanded or folded steel materials. It starts out as a planar sheet and is then formed by applied force. Examples are shown in Figure 27 (a,b). In (a) there are two large bowl-like forms. Note the difference between the large openings at the top due to stretching and the dense closed crumpled material at the base. This was a huge surprise to the artist.
Two examples of large bowl shapes with mirrored steel from the GR FLORA series are shown in Figure 28 (a,b). These forms are quite striking and are formed by force without the use of molds as in nature. These two are part of a series of five sculptures that have continually varying profiles and very different forms. A circular form is shown in (a) and a square form is shown in (b). As in (b) above, the spacing is large at the top and dense at the base.

![Figure 28. (a) Circular form, GR FLORA 24 100 2, mirrored stainless stel, 54 in dia, 2012, (b) Square form, GR FLORA 64 60 102, mirrored stainless steel, 54 in x 54 in, 2012.](image)

**X-Tower.**

Haresh Lalvani was invited to install two major works in the Purf series at the OMI International Arts Center in Ghent, NY in the summer of 2014. The X-Tower 88.2, from his tower series is shown in Figure 29 and is an installation at The Fields Sculpture Park. Here is a succinct quote from Haresh Lalvani: "Trees, natures anti-gravity inventions, stretch upwards to rise against gravity to shape themselves. Self-shaping is natures way to achieve its incredible forms. The X-TOWER series is born from this inspiration. It is dedicated to the majestic Sequoia, the tallest symbol of the diminishing plant kingdom which enables our planet (and us) to breathe."

There are several openings in X-TOWER 88.2 where the material has been stretched more than seems possible. One can enter the space of X-TOWER and look up as shown in the excellent view in Figure 30. One can hypersee X-TOWER from the inside as well as the outside. A detail image of an earlier smaller version in the series is shown in Figure 31, where one can see how the openings are formed. There are also interesting light and shadow effects when X-TOWER is seen in strong sunlight, and as light changes during the day.

**X-POD**

The second piece, an architectural sculpture Lalvani installed at OMI 2014, is the room size space X-POD 138 shown in Figures 31 and 32. X-POD 138 started out as a planar sheet just as X-Tower and force was applied to form the dome shape. Both pieces were installed on June 14, 2014 at Omi International Arts Center in Ghent, NY. The site for X-POD 138 is in Architecture Omi, the section of
Omi for experimental architecture. X-POD 138 has one space, one surface, and one opening exemplifying minimal architecture. This installation at Architecture Omi is the first architectural experiment using Milgo-Bufkin-Lalvani proprietary rapid forming method for making three-dimensional structures by expanding single sheets (metal, in this instance).

The Purf screen can be used to make relatively small items such as bowls and tables as well as larger sculptures and architectural structures. It is a very adaptable material. For emergency temporary housing, it could be shipped in sheets and then formed into enclosures. Lalvani is keen to build a 4-storyed tall X-Tower and a 20 ft wide X-Pod to explore the outer bounds of this process.

For further information on the works of Haresh Lalvani, we refer the reader to his website for a large collection of informative essays and videos available on Youtube.
Introduction

In this paper, I present recent sculptures of Sculptor Keizo Ushio from 2012 to 2014. The granite carvings of Keizo Ushio have previously been discussed in [1-7]. Note that all dimensions are as height, width, depth.


We begin with a sculpture that was commissioned by the City of Canberra, Australia, shown in Figures 1 and 2. This sculpture, called Oushi Zokei Dream Lenz for the Future, is a large divided torus with the drill holes rotated 180 degrees. This results in a space that is a half-twist Mobius band, or a "Mobius space". There are two types of contrasting surface treatment. One being bush hammered and the relatively rougher being large faceted regions. The setting is spacious with a background of trees.

Figure 1. Oushi Zokei Dream Lenz for the Future, 250 x 900 x 300 cm., White granite and natural stone base.

Figure 2. Darker color on a rainy day.

Oushi Zokei SxS, Cottesloe Beach, 2012

Keizo's 2012 SxS sculpture for Cottesloe is shown in Figures 3-5. SxS refers to Sculpture by the Sea. This sculpture consists of two parts. The gray part is one-half of a torus that was divided by drilling
holes that were rotated 360 degrees. This resulted in dividing the torus into two separate halves and then one half was removed. The darker stone was then placed about the torus.

Figures 3-5. SxS Oushi Zokei, Cottesloe Beach 2012, 226 x 260 x 100 cm., White and black granite, natural stone base; An alternate view; and sculpture with six friends.

Figures 6-7. SxS Oushi Zokei Bondi Beach 2012 Two Twist Bands, 216 x 132 x 68 cm., White granite.
Oushi Zokei SXS Bondi Beach, 2012

Here, SxS also refers to Sculpture by the Sea. For Oushi Zokai SxS Bondi, 2012, Keizo exhibited the divided double twist shown in Figures 6 and 7. The sculpture separates into two parts and the narrow space between the two parts is a double twist space. The dark shadows contrast well with the light gray color of the granite. Part of the surface is smooth and part is left in the natural state with quarry lines visible. The sculpture looks quite beautiful against the background of blue sea and sky.

Oushi Zokei SXS Cottesloe 2013

Keizo also created a divided double twist sculpture for Cottesloe, 2013 as shown in Figures 8 and 9. In this case the stone is black granite with three different contrasting surface treatments consisting of smooth, natural, and bush-hammered. Keizo encourages children to interact with his sculptures, as in Figure 9.

Figure 8. Oushi Zokei 2013, 230 x 135 x 100, Spanish Black Granite. Figure 9. She just reaches the top!

Oushi Zokei, SXS Bondi Beach, 2013.

Keizo's sculpture for Bondi, 2013 is shown in Figures 10 and 11. It is a torus divided into two interlocking halves, which is one of Keizo's most successful series. In this case the two halves are positioned in a new way that is diagonal instead of vertical and gives the sculpture a more active lean. The drill marks resulting from the process of dividing the stone are refined and they become a
A striking feature of the sculpture when enhanced by sunlight and shadow. The ocean and sky provide a perfect background.

Figure 10. Oushi Zokei, SxS Bondi Beach, 2013 Two Rings, 205 x 260 x 160 cm., Light brown granite.

Figure 11. Oushi Zokei, SxS Bondi Beach, 2013, Alternate view with Keizo.
Keizo was invited to have a solo exhibit in 2013 at the Himeji Gokoku Jinjya Shrine, where he exhibited five large sculptures, as shown in Figures 12-17.
Oushi Zokei, Private Garden Commission, Kobe, Japan, 2013.

The sculpture shown in Figure 18 is a classic Keizo divided Mobius. The polished brown granite works very well in the natural garden setting, where the placement allows for natural light. It is a beautiful image to contemplate.
Oushi Zokei, SXS Cottesloe, 2014.

The sculpture Keizo exhibited at Cottesloe 2014 was the completely new form shown in Figure 19. It is a divided triple twist Mobius. The division results in the sculpture forming a trefoil knot but not symmetric, which makes it even more interesting. Note that the position on the base is diagonal, when viewed from above, with the left (for viewer) front forward and right front back. This is a very dynamic position, similar to a person with a 45 degree torso twist with one shoulder forward and one back. Thus I also see it as a twisted torso and consider it one of Keizo’s finest works. He does keep coming up with impressive new ideas.

*Figure 19. Oushi Zokei Three Twist Heart, 205 x 100 x 80 cm., Green granite and natural stone base.*
Keizo’s outdoor entry for Sculpture by the Sea Bondi 2014 is shown in Figure 20. An innovation is the way the polished upper part seems to emerge from the natural lower part. It is a very impressive double twisted divided arch form. Keizo also showed five sculptures in the indoor exhibit. They are shown in Figures 21-25.

*Figure 20. Oushi Zokei Gift from the Earth Forest, Indian Galaxy granite and Spanish Black Granite, 2014, h 210 x w 220 x d 80 cm., private collection.*

*Figure 21. Oushi Zokei 2013-9, 2013, Japanese Green granite, 44 x 44 x 34 cm.*

*Figure 22. Oushi Zokei 2013-7, 2013, Japanese Gray granite, 46 x 37 x 20 cm.*

*Figure 23. Oushi Zokei 2013-19, 2013, African pure black granite, 47 x 50 x 20 cm.*
Eternity, Shikoku Island

In April, 2014 Keizo installed the sculpture *Eternity* at Symbol Road in the city of Nihama on Shikoku Island, Japan, shown in Figure 26. *Eternity* is a divided Mobius with refined drill marks. It was painted with natural Colcothar that contrasts nicely with the light brown color of the granite. *Eternity* has a unique appearance different from previous works.

References


Figure 27. Grandfather Keizo with his granddaughter Yuika Simizu, his pride and joy!
Introduction

A basic tetrahedron is shown in Figure 1(a) consisting of four equilateral triangles. The number of vertices \( V = 4 \), the number of edges \( E = 6 \), and the number of faces \( F = 4 \). Thus Euler’s formula yields \( V - E + F = 4 - 6 + 4 = 2 \) corresponding to a sphere.

First Stage of Generalization

We consider the four points (vertices) in Figure 1 as one pair corresponding to the endpoints of the upper edge and one pair corresponding to the endpoints of the opposite lower edge. The first generalization is to connect, by a straight line, each endpoint of these two edges to an arbitrary point on the opposite edge. For example, if both endpoints are connected to the midpoint of the opposite edge, then we obtain the generalized tetrahedron in Figure 1(b). First note that there are six vertices, three each on the upper and lower edges. There are 8 edges and four identical faces and
each face is bounded by four edges. We have $V - E + F = 6 - 8 + 4 = 2$ as above for a basic tetrahedron. The four faces are not planar. However, if a wire frame in the shape of a face was dipped in a soap solution, the resulting soap film minimal surface with the wire as boundary would be a hyperbolic surface. Thus we can consider Generalization 1 corresponding to a tetrahedron with four bounding hyperbolic surfaces.

For the next example, connect each endpoint to the corresponding point $1/3$ of the distance between points on the opposite edge. The resulting tetrahedron is shown in Figure 1(c). In this case there are four hyperbolic faces (surfaces), each bounded by five edges since there is an extra vertex in the middle of two edges on the same line. In this case the number of vertices is 8 and the number of edges is 10. Thus $V - E + F = 8 - 10 + 4 = 2$ as above.

**Second Stage of Generalization**

The second stage of generalizing a tetrahedron is to replace some or all straight edges by curves. If we replace the upper and lower edges by curves in Figure 1 (a), we obtain Figure 2.

![Figure 2](image)

![Figure 3](image)

![Figure 4](image)

![Figure 5](image)

![Figure 6](image)

![Figure 7](image)

**Figure 2-7. Curve-linear tetrahedron 1 with four conical faces and five variations.**

Note that there are four faces where each face has two straight edges and one curved edge. If we dipped a wire model of a face in a soap solution, there would be a soap film minimal surface that would be conical shaped. We can rotate a copy of Figure 2 a quarter turn and combine it with Figure 2 to obtain the variation in Figure 3. Note that Variation 1 is stable since it sits on four points. Variation 1 suggests an architectural application as a supporting tower. One way to add strength in Figure 2 is to apply Generalization 1 to Figure 2 by adding straight edges from the ends of the curves to the center point of the opposite curve, as in Figure 4. Lastly, a quarter turn rotation of Variation 2 can be combined with Variation 2 to obtain Variation 3 in Figure 5. This is Variation 1 with triangulation for added strength. Another way to add strength in Figure 2 is to insert a tetrahedron inside by connecting the points that divide the curved ends into thirds, as Figure 6. Variation 4 can be rotated a quarter turn and combined with itself as in Figure 7.
Groups

A larger support can be constructed by so-called grouping. A group corresponding to Figure 2 is shown in Figure 8. A group corresponding to Variation 2 in Figure 1 is shown in Figure 9.

Figure 8. Group 1.  Figure 9. Group 2.

Generalization 1.

We will now replace the upper and lower edges in Generalization 1 in Figure 1(b) by curves as in Figure 12. In this case there are six vertices, eight edges, and four hyperbolic faces, each bounded by four edges. Thus \( V - E + F = 6 - 8 + 4 = 2 \). By connecting 1/3 points, an inner tetrahedron is formed as in Figure 13(a). A quarter turn rotation of the tetrahedron in Figure 10 can be combined with itself, as in Figure 11(b). Four copies of Figure 12 and four copies of variation 7 can be grouped as in Figure 12 (a) and (b).

Figure 10.  Figure 11. (a-b) Variations 6 and 7.  Figure 12. (a-b) Groups 3 and 4.  Figure 10-12. Curve-linear tetrahedron 2 with four hyperbolic faces; its variations and groups.
Generalization 2.

We will now consider Generalization 2 in Figure 1(c) combined with a quarter turn rotation as in Figure 13. This variation could serve as a table support.

Next replace the upper and lower edges in Figure 1(c) by curves as in Figure 16(a). In this case there are \( V = 8 \) vertices, \( E = 10 \) edges, and \( F = 4 \) hyperbolic faces; hence \( V - E + F = 8 - 10 + 4 = 2 \). An inner tetrahedron can be formed by connecting the one-third points as in Figure 14 (b). The tetrahedron in Figure 14(a) can be combined with a quarter-turn rotation as in Figure 15(a). Variation 6 can be combined with a quarter-turn rotation as in (b). Groups corresponding to Figure 14(a) and variation 10 are shown in Figures 16(a) and (b) respectively.
Replacing all edges with curves

We shall now replace all the straight edges in Figure 1(b) with curves. We are going to replace the upper and lower edges with half circles and replace the other four edges with quarter circles. It is convenient to begin with Figure 17(a). For the circle from left to right, choose the lower half circle and the upper two concave quarter circles. For the other circle, front to back, choose the upper half circle and lower concave quarter circles. The resulting shape is shown in Figure 17(b). This shape is referred to as a femisphere and was discovered by the British woodworker J.Roberts many years ago. Here femi refers to the curved edges (Google, femisphere). A femisphere made of wire is a toy that rolls in a wobbly way. The half circles will leave tracks in sand that are spaced arcs. Two examples of solid wooden femispheres are shown in Figure 18. These were turned by Clyde Collier of the Gulf Coast Wood Turners Association, Houston, Texas.

Charles Perry (1929-2011), an eminent sculptor whose work was inspired by mathematics, independently discovered the femisphere by starting with a sphere that is then transformed by inverting the arcs of the surface on the lines of the stitching of a baseball. Perry referred to this shape as a mace and positioned a mace in a horizontal position as in Figure 18. Three Mace sculptures by Perry are shown in Figure 19. We note that both Roberts and Perry discovered the shape by considering a sphere rather than a tetrahedron.
Variations.

A quarter turn rotation of the femisphere is joined with itself in Figure 20. The center points of the quarter circles in Figure 17(b) can be joined by straight lines to form an inner linear tetrahedron, as in Figure 21. The outer femisphere can be considered as a curved tetrahedron. In general, the Mother and Child is a historical subject in sculpture, where the outer form protects the inner form. With this in mind, we can view the sculpture as a Curved Mother and Linear Child.

We shall now consider some further embellishments of Figure 17(b). The midpoints of the convex half circles in (b) can be connected by straight lines to obtain a second inner linear tetrahedron, as in Figure 22.

If the points one-third of the way along the convex quarter circles in Figure 9 are connected by straight lines, then a second taller inner linear tetrahedron is obtained as in Figure 23.

If the structure in Figure 22 is rotated a quarter turn and joined with the original structure, than the form in Figure 24 is obtained.

If we only connect the midpoints of the convex quarter circles in Figure 17 (b), than we obtain Figure 22 without the inner tetrahedron, as in Figure 25. Variation 17 is now rotated a quarter turn and combined with itself to obtain Variation 18 in Figure 26. For additional variations, we can embellish Figure 17(a) as shown in Figures 27 – 29.
Additional Versions of Figure 1(a).

We will now consider further modifications of the basic tetrahedron Figure 1(a). First replace the diagonal edges by convex curves, as in Figure 30(a). Join a quarter turn rotation of this figure with itself, as in (b). A corresponding group is shown in (c).
The next modification is to replace the upper and lower edges in Figure 30(a) by concave curves as in Figure 31(a). A quarter turn joining and a group are shown in (b) and (c) respectively.

Next all edges in Figure 1(b) are replaced by concave curves, as in Figure 32(a). A quarter turn of (a) is joined to (a) in (b) and a group is shown in (c).

Figure 31. (a) Curved Tetrahedron 3 with four hyperbolic faces. (b) Quarter turn rotation of (a) joined with (a). (c) Group corresponding to (a).

Figure 32. (a) Curved Tetrahedron 6 with four hyperbolic faces. (b) Quarter turn rotation of (a) joined with (a). (c) Group corresponding to (a).
Introduction

In this paper, I present a series of paintings by Benigna Chilla, which were exhibited at Tibet House in New York City, May 23- August 1, 2014. Three gallery views at Tibet House are shown below in Figures 1-3. The series of paintings will be exhibited at the Rochester Institute of Technology, September-November, 2015.

**Benigna Chilla’s Statement**

**Figure 1. A View of the Tibet House Gallery.**

**Figure 2. Another Gallery View of Tibet House.**

**Figure 3. An additional View of the Gallery.**
Selection of Works

The paintings in Figures 1 and 2 are symmetric about the vertical center line. In Figure 3, the painting on the left is symmetric about the vertical center line and the painting on the right is asymmetric. There is a strong use of black which brings out the intensity of a striking variety of color combinations. Texture is imbedded onto the canvas as there are several layers of paint in each piece.

The painting in Figure 4 combines a central geometric pattern bounded above and below by the black forms. This pattern is symmetric about the central horizontal and vertical lines. Each black shape has two straight sides and two contrasting curved sides. The straight sides also form inner boundaries of yellow shapes with contrasting outer curved boundaries. This central geometric pattern is relatively simple compared to the more detailed surrounding consisting, for example, of a pattern of leaves above and repeated small spirals on the sides. The upper and lower black curves are bounded by arc shapes and the arc shapes are bounded by triangular shapes of the same light color above and three different colors below. After having concentrated on this image for some time, the black shapes seemed to lift off the surface.

The painting in Figure 5 has a basic upper square with a central black octagon that stands out against the contrasting delicately colored overlapping disks. As in Figure 4, there are also rows of varying design below the square including a narrow row of black and white crosses.

In Figure 6, there is a central square with a geometric black pattern that stands out against an orange background. The black pattern frames a gray disk with a red circular border, each with detailed designs. There are rows of varying designs above and below the central square. In particular, there is a row of whirling black and white spirals above that contrasts with the central square.
The painting in Figure 7 has a strong central black trapezoid standing out against a yellow background with subtle small circular designs. The trapezoid has a contrasting checkered rectangular above and a contrasting diamond pattern rectangle below. At the top there is a row of small white circles below a row of five colored disks balancing the rows of diamonds below.

The central outer rectangle in Figure 8 encloses a symmetric structure consisting of an equilateral black triangle balanced on point on a second equilateral black triangle. The upper triangle has supporting patterned rectangles on each side resting on yellow triangles. This structure is surrounded by a varying detailed background. There are contrasting rows of repeating designs above and below the main rectangle.
The painting in Figure 9 has a central outer rectangle bounded on each side by narrow red rectangles and blue strips. These verticals contain a square with a repeating pattern of gray corners, red diamond, black corners, red square, gray corners, black corners, and a gray square with eight small octagons. This repeating central pattern is more complicated than the central patterns in Figures 4-8. There are detailed rectangular borders above and below the central rectangle.

Additional paintings of Chilla are shown Figure 10-14. They are symmetric about the vertical center line. Figure 14 is also symmetric about the horizontal center line. In particular, there are symmetric black shapes on the vertical center line that contrast with adjacent shapes with detailed lightly colored patterns.

For additional works by Benigna Chilla, visit her website www.benignachilla.com.
This Bow-Tie Tetra-Tangle was constructed specifically for the Exhibition of Mathematical Art to be held at the Joint Mathematics Meeting (JMM) in San Antonio, TX, in January 2015. It resulted from an extension of the work on “LEGO® Knots” reported at Bridges Seoul Conference in August 2014. In that project, a small set of snap-together, tubular parts had been developed to permit the construction of sculptural maquettes in the form of prismatic extrusions along modular space curves composed of circular arcs.

Figure 1: “Tetra-Tangle of Four Bow-Tie Links” by Carlo H. Séquin, 9in × 9in × 9in.
The concept of such modular components was inspired by Henk van Putten’s “Borsalino” sculpture [3] presented at Bridges 2013 in Enschede. It is composed of two types of modules (Fig. 2a): the three (brown) end-caps that form tight 180° turns, and the six (green and cyan) connector pieces, which exhibit gentler bends through an angle of 45°. Henk van Putten’s sculptures, while modular in their geometry, are constructed as coherent solid objects. But I thought it would be a lot of fun, if one could play with such geometrical building blocks in real time in a tangible manner. So I constructed some LEGO®-like, snap-together parts with inexpensive layered-manufacturing techniques. For most of my experiments I used Fused Deposition Modeling (FDM) machines from Stratasys. Figure 2b shows a realization of the “Borsalino” shape from nine plastic parts built on such a machine.

Playing with the initial set of parts was indeed very inspiring, and I soon added additional modules into my building-block set: curved connector pieces and end-caps with different bending radii. During these experiments, it occurred to me that stretching the connection in the middle of each curved connector pair (Fig. 2c) would lead to another interesting configuration. When the inserted tube is of just the right length, the two square end-cross-sections that previously were connected by an end-cap shift past one another until they are located corner to corner (Fig. 2c). Now, to close off the two open ends, we need a new end-cap that sweeps the square cross section through a half-circle around one vertex, parallel to one of its face diagonals. This is the same as sweeping a “rhombic” cross section, i.e., a square with an azimuthal rotation of 45°, along an arc with a radius enlarged by $\sqrt{2}$. This rhombic end-cap together with two attached curved connector pieces forms an interesting sculptural “Bow-Tie” shape. Figure 3c shows the resulting “Bow-Tie Borsalino.”

![Figure 2: Borsalinos: (a) 9-part geometry by Henk van Putten; (b) realization with FDM; (c) extension pieces inserted between curved connectors; (d) resulting “Bow-Tie Borsalino.”](image)

![Figure 3: (a) Borsalino shape with triangular cross-section; (b) a flush tangle of three prismatic beams cut to proper length; (c) Bow-Tie-Loop formed with three helically twisted end-caps;](image)
In a different set of experiments with Michelle Galemmo we investigated what happens when the square cross-section in these “Borsalino” shapes is replaced with an equilateral triangle [1]. In addition to some “Borsalino” shapes with twisted arms (Fig.3a), a variety of different “Bow-Tie Loops” can be formed. We start by placing three prismatic prisms flush against one another (Fig.3b) and then truncate their length to the point where their outer edges intersect. Wherever a pair of end-faces of these prisms share a vertex, we add a specially constructed, “helically” twisting end-cap. The result is a single, closed sweep of the given triangular cross section, forming a 3-lobe Bow-Tie loop (Fig.3c).

![Figure 4: More Bow-Tie Loops: (a) three lobes with a pentagonal beam, (b) the six parts used in its construction; (c) four-lobe Bow-Tie Loop with a triangular cross section.](image)

There are more ways to make such Bow-Tie Loops. Figures 2d and 3c show ways to make 3-lobe Bow-Tie Loops with beams that have regular quadrilaterals and triangular cross sections, respectively. Figure 4a demonstrates that this can also be done with a beam with a pentagonal cross-section. Figure 4b shows the six plastic parts that were used in its construction. Alternatively, one can pack up to five triangular prisms tightly and symmetrically around the origin so that each beam is in flush face-to-face contact with its nearest two neighbors. Again the beams are truncated to the point where their outer edges intersect, and neighbors are then joined with custom-made helical end-caps. Figure 4c shows a 4-lobe realization.

In the implementation of these various Bow-Tie Loops, I experimented with different ways of partitioning and modularizing the beam. For the shapes shown in Figures 3c and 4a, the beam was split into six parts. Figure 4c shows an assembly of only four parts, where each one of them is a Bow-Tie lobe starting and ending at the origin. It results in nice smooth lobes – but it was very difficult to assemble!
With the sculpture "Tetra-Tangle of Four Bow-Tie Links" (Fig.1), constructed for JMM 2015, I tackled the problem how several such Bow-Tie Loops might be mutually interlinked (Fig.5b). The key inspiration came from the TETRAXIS® puzzle, which consists of four sets of three mutually parallel, 3-sided prisms, pointing in four different tetrahedral directions (Fig.5a). There are eight points where three beams join in a flush, symmetric tangle, and six points with tight, 4-beam tangles. When two triangular prism-end-faces that share a common vertex are closed off with a connecting sweep, a loose "Bow-Tie" is formed. If all twelve pairs of adjoining triangular end-faces are connected in this way, the result is a link of four mutually interlocking, 3-sided Bow-Tie Loops. This represents an alternating 12-crossing link that has the same connectivity as the "Tetra-Tangle," which I constructed from 4"-diameter card-board tubes in 1983 (Fig.5c). The new geometry has been realized as four differently colored loops, each composed of six tubular snap-together parts (Fig.6) fabricated on an FDM machine. The overall assembly takes place one ring at a time. For the incorporation of the fourth loop, first the straight segments are introduced into the partial assembly (Fig.7a) and then the twisted end-caps are applied. The last end-cap to be applied took a little extra force, since its two insertion sleeves are not parallel.

References:
Figure 7: Assembly of the Bow-Tie Tetra-Tangle: (a) 3 loops assembled; (b) all 4 loops assembled.
Cover Sculpture: Oushi Zokei; Three Twist Heart by Ushio Keizo